AVERAGE AND PROBABILITY.

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NOTE ON PROBLEM 26, AVERAGE AND PROBABILITY.

BY G. B. M. ZERR.

In reply to Dr. Martin, for whom I have the utmost respect, I have the following remarks to make. The problem that gives the result $\frac{1}{6}a^2$ is different from the problem that gives the result $\frac{a^2}{2\pi}$. In the former the right angle remains fixed and does not lie on a circle as Dr. Martin states. The problem is as follows: Find the average area of all triangles formed by a straight line of constant length a sliding so that its extremities constantly touch two fixed straight lines at right angles to one another. In the problem under consideration the hypotenuse a is fixed and the right angle moves on the semi-circumference. In the first case the average length of one leg is $\int_0^a xdx \Big/ \int_0^a dx = \frac{1}{2}a$. In the second case the average length is $\int_0^a xds \Big/ \int_0^a ds = a/\pi$. In the first case the average area of all the triangles is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}dx \Big/ \int_0^a dx = \frac{1}{6}a^2$. In the second case the average area is $\int_0^a \frac{1}{2}x\sqrt{a^2-x^2}ds \Big/ \int_0^a ds = \frac{a^2}{2\pi}$, where ds represents an element of arc. It is plainly evident that in the result $\frac{1}{6}a^2$ the leg does not and cannot change its direction or its average length would not be $\frac{1}{2}a$. In the second case it is constantly changing its direction and the right angle is moving on a semicircumference. The problem calls for a given hypotenuse and not one that is constantly changing its direction; hence the result $\frac{a^2}{2\pi}$ is the correct result.

DR. MARTIN'S RESULT IS NOT CORRECT.

F. P. MATZ.

Cause the problem to read: "Find the average area of all right-angled triangles having a given hypotenuse, *if an arm of the triangle vary uniformly;*" then Dr. Martin's result, $\frac{1}{6}h^2$, is perfectly correct.

Strip the problem of this italicised condition; that is, make the problem read as originally proposed; then the number of possible right-angled triangles is proportional to the *length* of the semicircumference of which the given hypotenuse is the diameter. This is *the correct* plan of solution. By *adhering* to *this* plan of solution, the correct result, $h^2/2\pi$, is obtained, regardless as to choice of independent variable.

Dr. Martin's result, $\frac{1}{6}h^2$, is *too great;* for he, by making the number of possible right-angled triangles "proportional to the given hypotenuse," *ignores*

the consideration of the areas of practically an infinitude of right-angled triangles of which the major portion have one *rather small* acute angle—thus giving them areas *smaller* than $\frac{1}{8}h^2$.

Since not only all of Dr. Martin's *ignored* right-angled triangles, but *all possible* right-angled triangles, have been properly averaged in my solutions leading to (the result) $h^2/2\pi$, I repeat that this result is the correct one.

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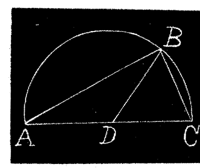
A REPLY TO DR. MARTIN'S NOTE.

BY THE EDITOR OF THIS DEPARTMENT.

I will say at first, that I too, have profound regard for Dr. Martin, and his opinion on a subject in which he was the pioneer writer in America should not be assailed simply for the sake of controversy.

His argument is entirely sound as to fact but not as to interpretation. It is true the triangles are *not* uniformly distributed on the semicircumference IF the number of triangles is to be obtained by varying one of the legs of the triangle. That this is true may be easily shown from the figure. Let $AC=a$, the hypotenuse, and $BC=x$, angle $BDC=\theta$. Then $x=a\sin\angle BDC=a\sin\frac{1}{2}\theta$. Differentiating, we have $\frac{dx}{d\theta}=\frac{1}{2}a\cos\frac{1}{2}\theta=\frac{1}{2}a\sqrt{\frac{1}{2}(\cos\theta+1)}$. $\therefore dx$ increases $\sqrt{\frac{1}{2}(\cos\theta+1)}$ times as fast as $d\theta$. When $\theta=0$, dx and $d\theta$ are increasing equally, and when $\theta=\frac{1}{2}\pi$, dx is increasing $\frac{1}{2}\sqrt{2}$ times as fast as $d\theta$. Hence it is evident that a greater number of triangles exist for a certain length of arc in the vicinity of the vertex of the semicircumference whose diameter is the hypotenuse, than for the same length of arc near the origin C when the number of triangles is made a function of one of the legs of the triangle, and therefore Dr. Martin's conclusion is sound if we grant his assumption, namely, that the number of triangles is a function of one of the legs of the triangle.



But this assumption is what we refuse to grant. We believe that there are other triangles that are to be interpolated in the series in order that the totality of the triangles may be obtained and that these interpolated triangles are found by making the totality a function of the semicircumference.

From this consideration, it is evident that Dr. Martin's result, $\frac{1}{8}a^2$, is greater than the result, $\frac{a^2}{2\pi}$, which we are defending. The reason is, that according to his interpretation, the triangles are most numerous when $x=\frac{1}{2}\sqrt{2}a$, that is to say, when the vertex of the triangle coincides with the vertex of the semicircumference. Hence the sum of the areas of the triangles ought to be greater than when only as many triangles are taken in one portion of the arc of the semicircumference as in any other.

If the radius DB is made to revolve with uniform velocity about the point